# Instabilities Driven Equilibration at the Early Stage of Nuclear Collisions

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Evidence of the early stage equilibration

Success of hydrodynamic models in describing elliptic flow
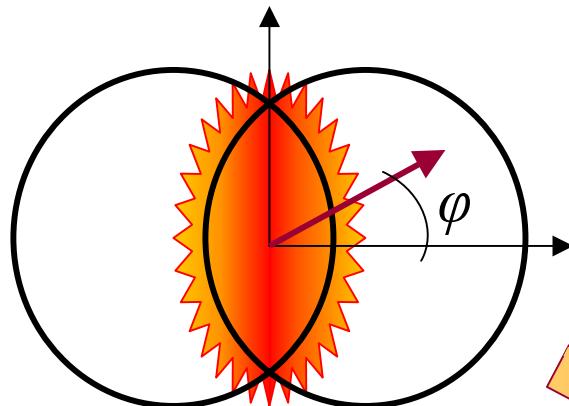
Hydrodynamics

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla \right)\mathbf{v} = -\frac{\nabla p}{\rho}$$

Hydrodynamic requires local thermodynamical equilibrium!
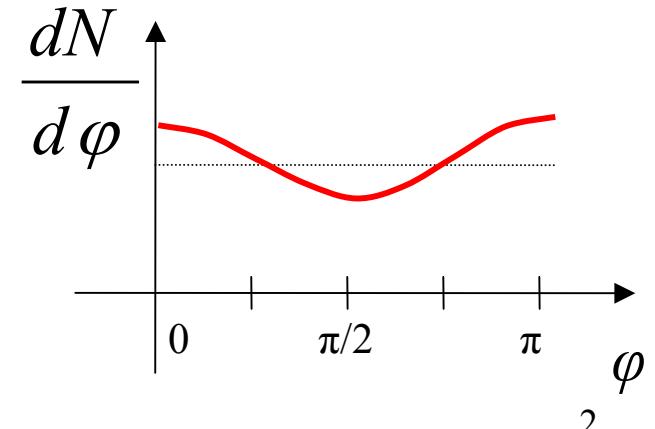
Equilibration is fast

$$v_2 \sim \varepsilon = \left\langle \frac{x^2 - y^2}{x^2 + y^2} \right\rangle$$

Eccentricity decays due to the free streaming!

$$\varepsilon \searrow \Rightarrow v_2 \searrow \qquad \Longrightarrow \qquad t_{eq} \leq 0.6 \ \text{fm}/c$$

time of equilibration

U. Heinz, AIP Conf. Proc.**739,** 163 (2004)

Collisions are too slow

Time scale of hard parton-parton scattering

hard scattering ~ momentum transfer of order of T

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g)T}$$

either single hard scattering or multiple soft scatterings

$$t_{\text{eq}} \approx t_{\text{hard}} \geq 2.6 \ \text{fm}/c$$

R. Baier, A.H. Mueller, D. Schiff & D.T. Son, Phys. Lett. **B539**, 46 (2002)

Instabilities

$$A(t) = A_0 + \delta A(t)$$

fluctuation

Instability

$$\delta A(t) \propto e^{\gamma t}$$

$$\gamma > 0$$

stable configuration

A_0

$A(t)$

unstable configuration

A_0

$A(t)$

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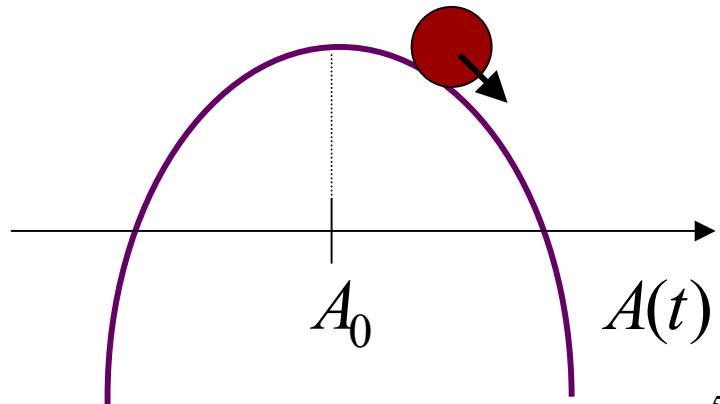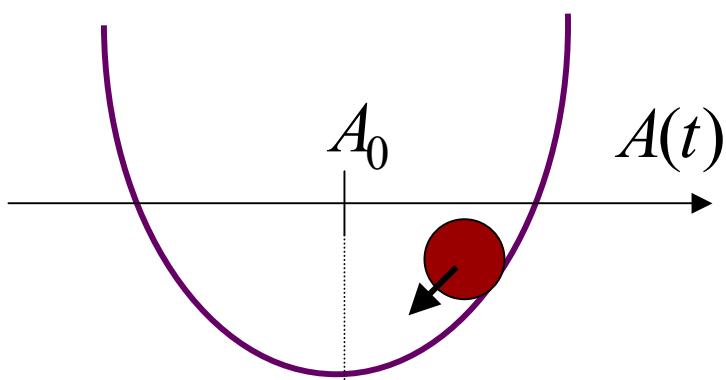
Plasma instabilities

▶ **instabilities in configuration space – hydrodynamic instabilities**

▶ **instabilities in momentum space – kinetic instabilities**

instabilities due to non-equilibrium momentum distribution

$$f(\mathbf{p}) \text{ is not } \sim \exp\left(-\frac{E}{T}\right)$$

Kinetic instabilities

▶ **longitudinal modes** – $\mathbf{k} \parallel \mathbf{E}$, $\delta\rho \sim e^{-i(\omega t - \mathbf{k}\mathbf{r})}$

▶ **transverse modes** – $\mathbf{k} \perp \mathbf{E}$, $\delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{k}\mathbf{r})}$

$\mathbf{E}$ – electric field, $\mathbf{k}$ – wave vector, ρ – charge density, $\mathbf{j}$ - current

unstable configuration

plasma

$f(p_x, p_y, p_z)$

beam

0   p_x
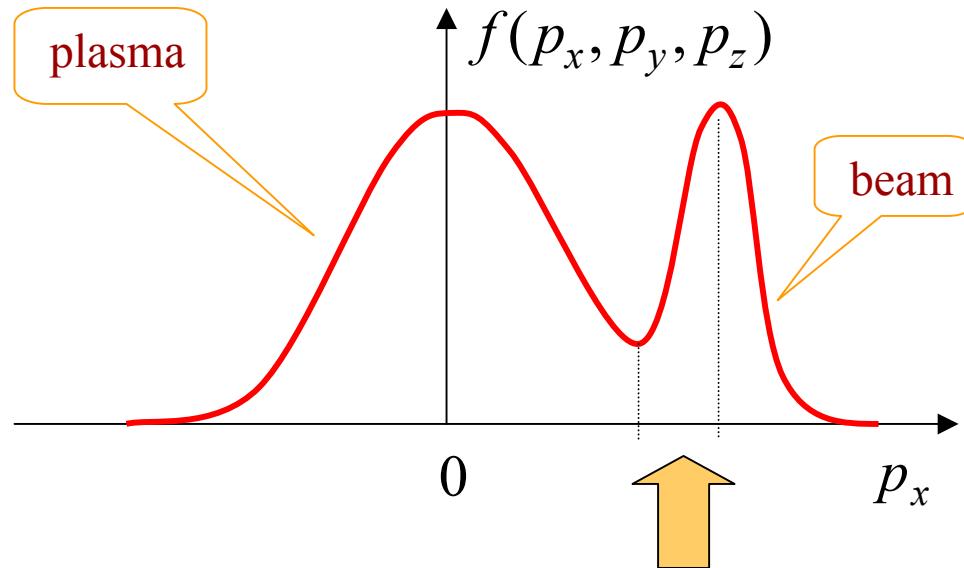
Energy is transferred from particles to fields

Logitudinal modes

Electric field decays - damping

$$f(p_x, p_y, p_z)$$

$$0 \qquad \frac{\omega}{k_x} \qquad \frac{p_x}{E}$$

particle acceleration

particle deceleration
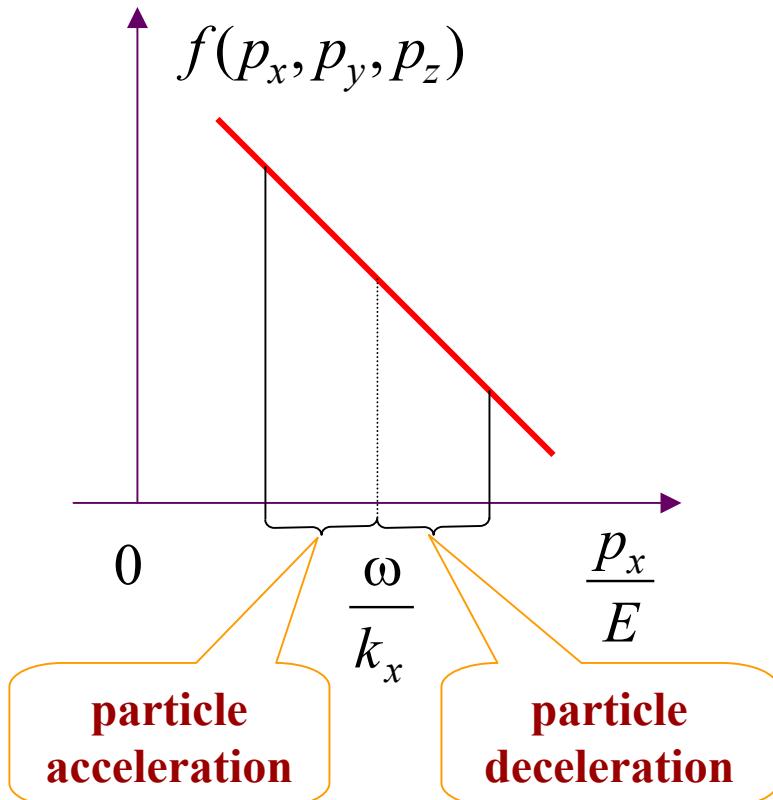
Electric field grows - instability

$$f(p_x, p_y, p_z)$$

$$0 \qquad \frac{\omega}{k_x} \qquad \frac{p_x}{E}$$

particle acceleration

particle deceleration
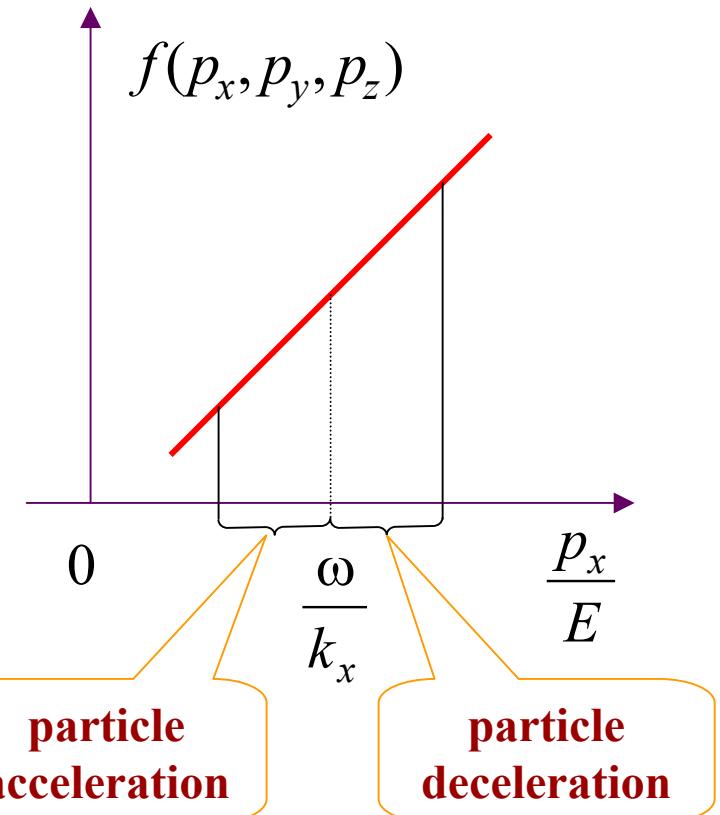
$\dfrac{\omega}{k_x}$ - phase velocity of the electric field wave, $\dfrac{p_x}{E}$ - particle's velocity   9

Unstable modes occur due to anisotropy of the momentum distribution

$f(\mathbf{p}) = f(-\mathbf{p})$
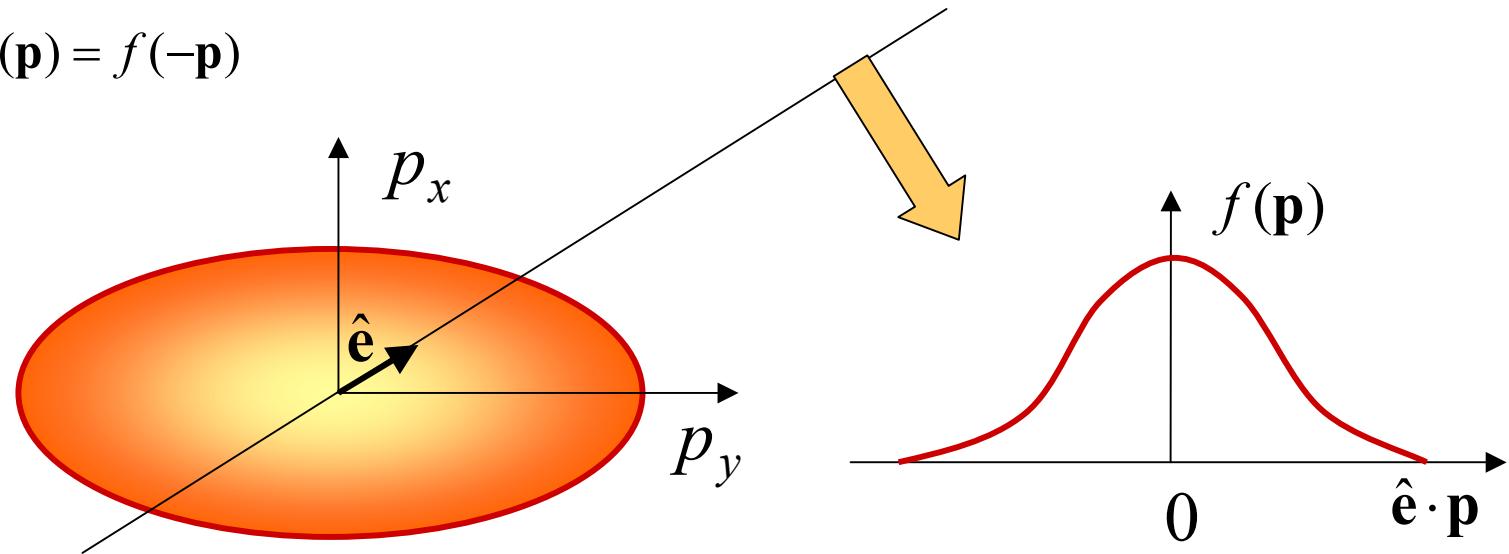
Momentum distribution distribution can monotonously decrease in every direction

Transverse modes are relevant for relativistic nuclear collisions!

Momentum Space Anisotropy in Nuclear Collisions

**Parton momentum distribution is initially strongly anisotropic**

p_T

p_L

time

p_T

p_L
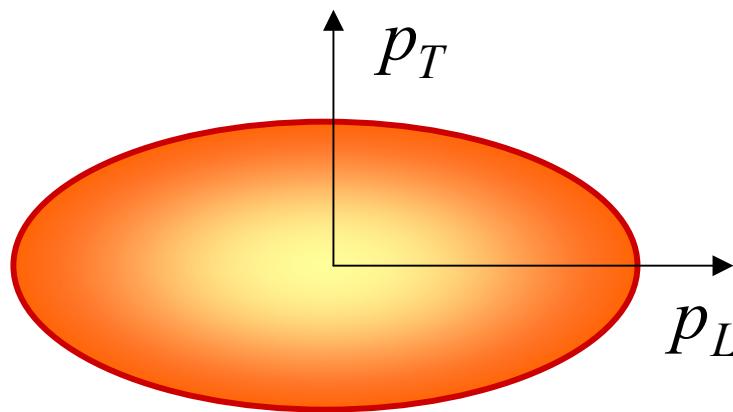
CM after 1-st collisions
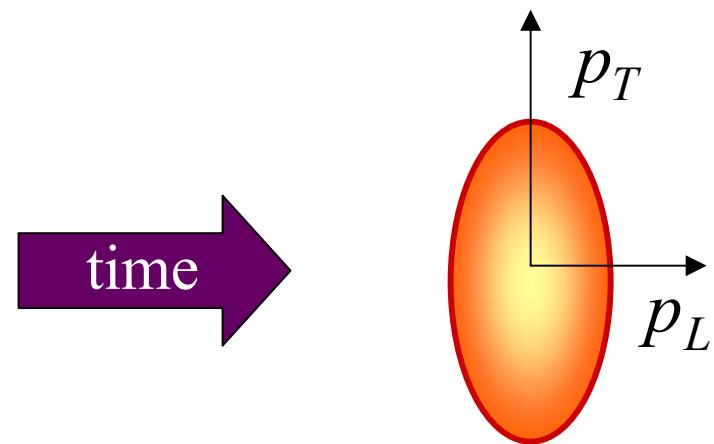
local rest frame

$$\left\langle j_a^\mu(x) \right\rangle = 0 \quad \text{but current fluctuations are finite}$$

$$\left\langle j_a^\mu(x_1)\, j_b^\nu(x_2) \right\rangle = \frac{1}{8}\delta^{ab}\int \frac{d^3p}{(2\pi)^3}\frac{p^\mu p^\nu}{E_p^2}\, f(\mathbf{p})\delta^{(3)}(\mathbf{x}-\mathbf{v}t) \neq 0$$

$$x_1 = (t_1, \mathbf{x}_1), \quad x_2 = (t_2, \mathbf{x}_2), \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$
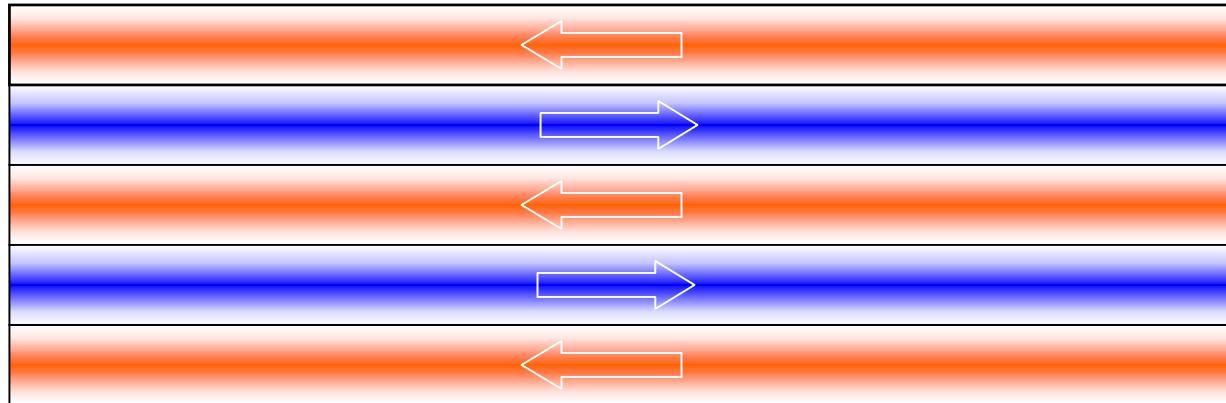
Direction of the momentum surplus

Mechanism of filamentation
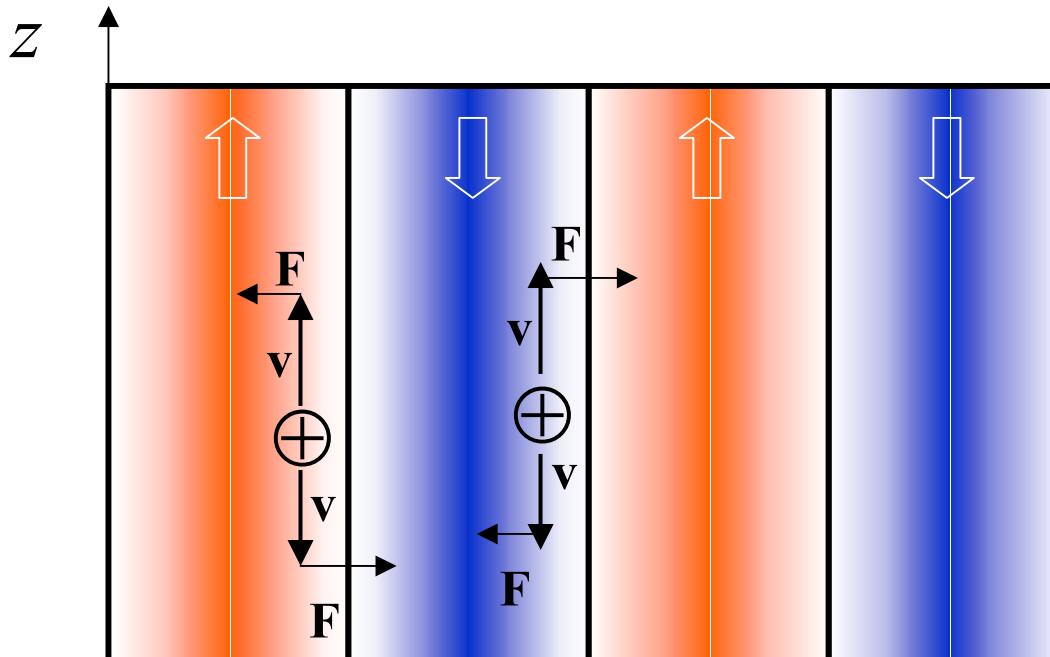
Lorentz force

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

Ampere's law

$$\nabla \times \mathbf{B} = \mathbf{j}$$

j_z

B_y

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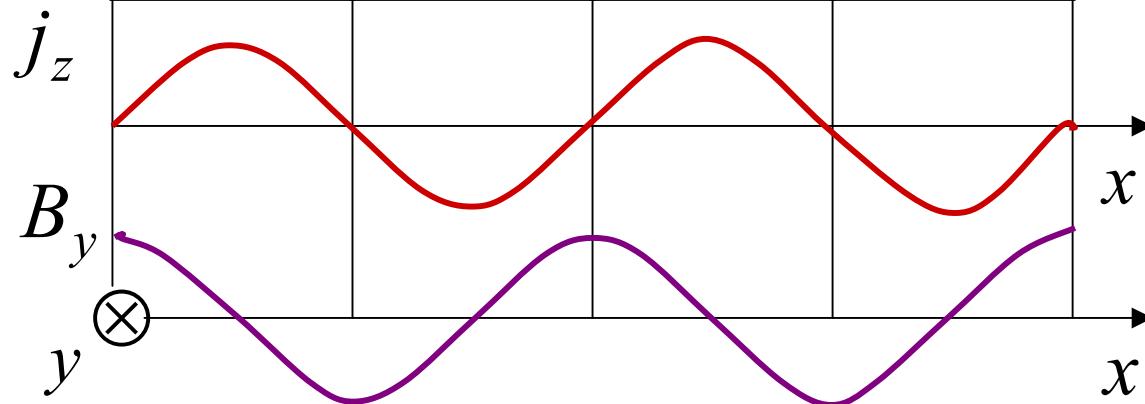
Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

gluon self-energy

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

Instabilities – solutions with $\mathrm{Im}\,\omega > 0$ $\quad \Rightarrow \quad A^\mu(x) \sim e^{\mathrm{Im}\,\omega\, t}$

Dynamical information is hidden in $\Pi^{\mu\nu}(k)$. **How to get it?**

Transport theory – transport equations

fundamental

$$\left(p_\mu D^\mu - g p^\mu F_{\mu\nu}(x)\partial_p^\nu\right)Q(p,x) = C \qquad \text{quarks}$$

$$\left(p_\mu D^\mu + g p^\mu F_{\mu\nu}(x)\partial_p^\nu\right)\overline{Q}(p,x) = \overline{C} \qquad \text{antiquarks}$$

adjoint

$$\left(p_\mu \mathcal{D}^\mu - g p^\mu \mathcal{F}_{\mu\nu}(x)\partial_p^\nu\right)G(p,x) = C_g \qquad \text{gluons}$$

free streaming   mean-field force   collisions

$$D^\mu \equiv \partial^\mu - ig[A^\mu, ...], \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$D_\mu F^{\mu\nu} = j^\nu[Q,\overline{Q},G]$$ mean-field generation

collisionless limit: $C = \overline{C} = C_g = 0$

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fluctuation

$$Q(p,x) = Q_0(p) + \delta Q(p,x)$$

stationary colorless state $\quad Q_0^{ij}(p) = \delta^{ij} n(p)$

$$|Q_0(p)| >> |\delta Q(p,x)|, \quad |\partial_p^\mu Q_0(p)| >> |\partial_p^\mu \delta Q(p,x)|$$

Linearized transport equations

$$p_\mu D^\mu \delta Q(p,x) - gp^\mu F_{\mu v}(x) \partial_p^v Q_0(p) = 0$$

$$p_\mu D^\mu \delta \overline{Q}(p,x) + gp^\mu F_{\mu v}(x) \partial_p^v \overline{Q}_0(p) = 0$$

$$p_\mu \mathcal{D}^\mu \delta G(p,x) - gp^\mu \mathcal{F}_{\mu v}(x) \partial_p^v QG_0(p) = 0$$

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Transport theory – polarization tensor

$$\delta Q(p,x) = g \int d^4 x' \Delta_p(x-x') \, p^\mu F_{\mu\nu}(x) \partial_p^\nu Q_0(p)$$
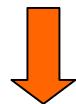
$$j^\mu[\delta Q, \delta\overline{Q}, \delta G]$$

$$p_\mu D^\mu \Delta_p(x) = \delta^{(4)}(x)$$

$$j^\mu(k) = \Pi^{\mu\nu}(k) A_\nu(k)$$

$$f(\mathbf{p}) \equiv n(\mathbf{p}) + \overline{n}(\mathbf{p}) + 2n_g(\mathbf{p})$$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu}{E} \Big[g^{\mu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+} \Big] \frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

Diagrammatic Hard Loop approach

$$\Pi^{\mu\nu}(k) = \left[\quad + \quad + \quad \right]$$

Hard loop approximation: $k^{\mu} << p^{\mu}$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^{\mu}}{E} \left[g^{\mu\lambda} - \frac{p^{\nu} k^{\lambda}}{p^{\sigma} k_{\sigma} + i0^{+}} \right] \frac{\partial f(\mathbf{p})}{\partial p^{\lambda}}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_{\mu} \Pi^{\mu\nu}(k) = 0$$

Dispersion equation

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(k) \qquad \textbf{chromodielectric tensor}$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

Dispersion equation

$$\det[\mathbf{k}^2 \delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} + \frac{g^2}{2\omega}\int\frac{d^3p}{(2\pi)^3}\frac{v^i}{\omega - \mathbf{k}\mathbf{v} + i0^+}\frac{\partial f(\mathbf{p})}{\partial p^l}\left[\left(1 - \frac{\mathbf{k}\mathbf{v}}{\omega}\right)\delta^{lj} + \frac{k^l v^j}{\omega}\right]$$

$$\mathbf{v} \equiv \mathbf{p}/E$$

Dispersion equation – configuration of interest

y ⊗ — z →

x

Direction of the momentum surplus
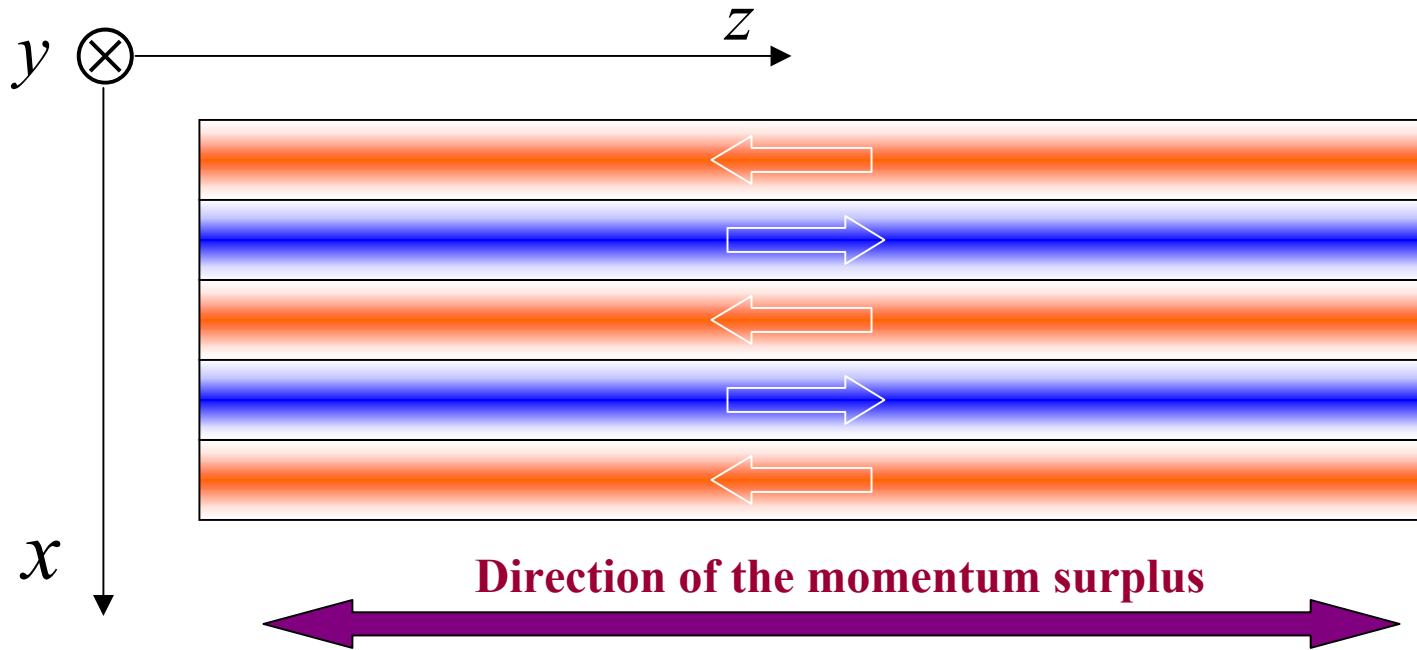
$$\mathbf{j} = (0,0,j), \quad \mathbf{E} = (0,0,E), \quad \mathbf{k} = (k,0,0)$$

Dispersion equation
$$k^2 - \omega^2 \varepsilon^{zz}(\omega,k) = 0$$

$$H(\omega) \equiv k^2 - \omega^2 \varepsilon^{zz}(\omega, k)$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{1}{H(\omega)} \frac{dH(\omega)}{d\omega} = \begin{cases} \oint_C \frac{d\omega}{2\pi i} \frac{d \ln H(\omega)}{d\omega} = \ln H(\omega)\Big|_{\phi=\pi^+}^{\phi=\pi^-} \\ \text{number of zeros of } H(\omega) \text{ in } C \end{cases}$$
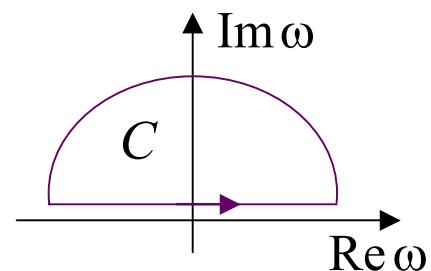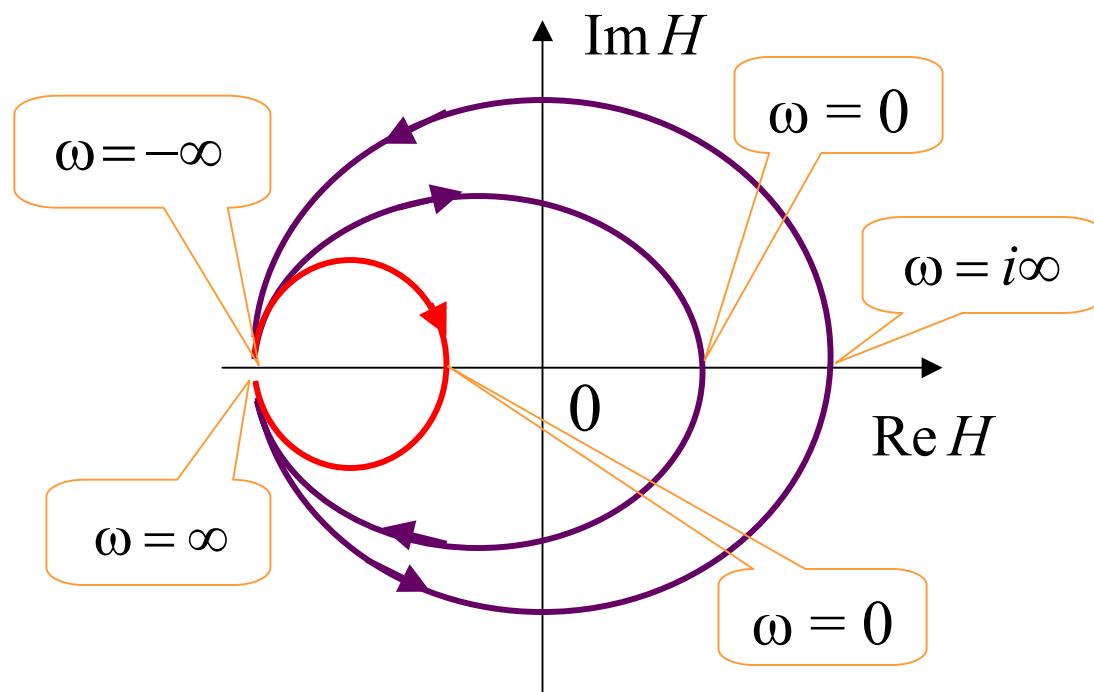
There are unstable modes if

$$H(\omega = 0) < 0$$

Anisotropy!

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Unstable solutions

$$f(\mathbf{p}) = \frac{2^{1/2}}{\pi^{3/2}} \frac{\rho \sigma_\perp^4}{\sigma_\parallel} \frac{1}{(p_\perp^2 + \sigma_\perp^2)^3} e^{-\frac{p_\parallel^2}{2\sigma_\parallel^2}}$$

$\rho = 6 \text{ fm}^{-3}$

$\alpha_s = g^2 / 4\pi = 0.3$

$\sigma_\perp = 0.3 \text{ GeV}$

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

solution

$$\omega(k) = \pm i \, \gamma_k$$

$$0 < \gamma_k \in \Re$$

pQCD profile
- - - - $\sigma_\parallel = 3.0$ GeV
— — $\sigma_\parallel = 2.0$ GeV
——— $\sigma_\parallel = 1.0$ GeV
—·—· $\sigma_\parallel = 0.5$ GeV

Growth rate γ_k (GeV) vs Wave number k_T (GeV)

J. Randrup & St. M., Phys. Rev. C **68**, 034909 (2003)

SU(2) Hard Loop Dynamics

1+1 dimensions

$$A_a^\mu = A_a^\mu(t,z)$$

Scaled field energy density

Anisotropic particle's momentum distribution

$$f(\mathbf{p}) = f_{\text{iso}}(|\mathbf{p}| + \zeta p_z)$$

$$m_D^2 = -\frac{\alpha_s}{\pi} \int_0^\infty dp\, p^2 \frac{df_{\text{iso}}(p)}{dp}$$

$$(m_D, \zeta)$$

total   **transverse magnetic**

Legend:
- $\mathcal{E}(E_z) / (m_\infty^4/g^2)$
- $\mathcal{E}(E_T) / (m_\infty^4/g^2)$
- $\mathcal{E}(B_T) / (m_\infty^4/g^2)$
- $\mathcal{E}(B_z) / (m_\infty^4/g^2)$
- $\mathcal{E}(HL) / (m_\infty^4/g^2)$

x-axis: $\gamma_* t$

Strong anisotropy $\zeta = 10$ γ_* - maximal growth rate

A. Rebhan, P. Romatschke & M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005)

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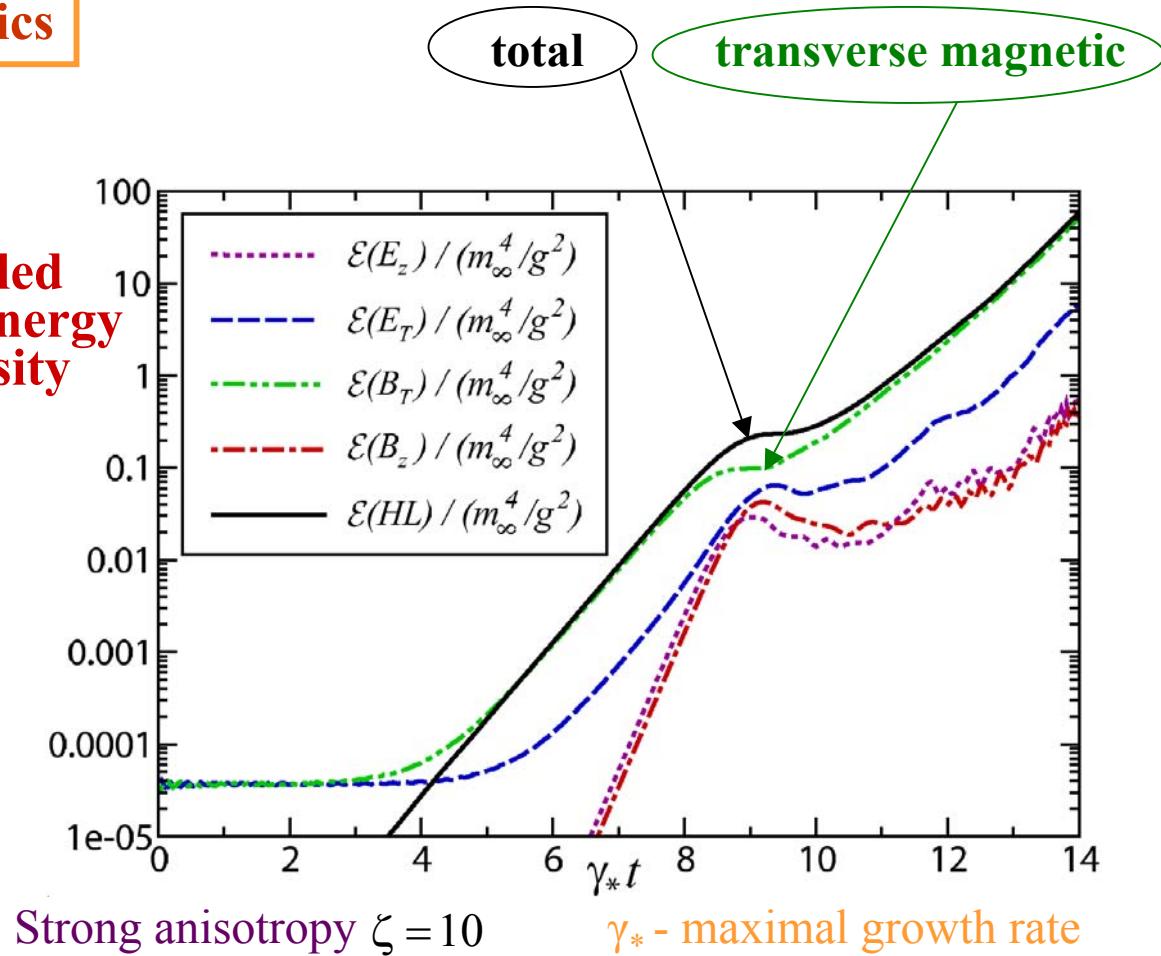
Growth of instabilities – 1+1 numerical simulations

**Classical system of colored particles & fields**

initial fields: Gaussian noise as in Color Glass Condensate

initial anisotropic particle distribution:

$$f_0(\mathbf{p}, \mathbf{x}) \sim \delta(p_x)\, e^{-\frac{\sqrt{p_y^2 + p_z^2}}{p_{\text{hard}}}}$$

$$p_{\text{hard}} = 10\ \text{GeV}$$

$$L = 40\ \text{fm} \qquad \rho = 10\ \text{fm}^{-3}$$

A. Dumitru & Y. Nara, hep-ph/0503121

Abelian 1+3

nonAbelian 1+1

nonAbelian 1+3
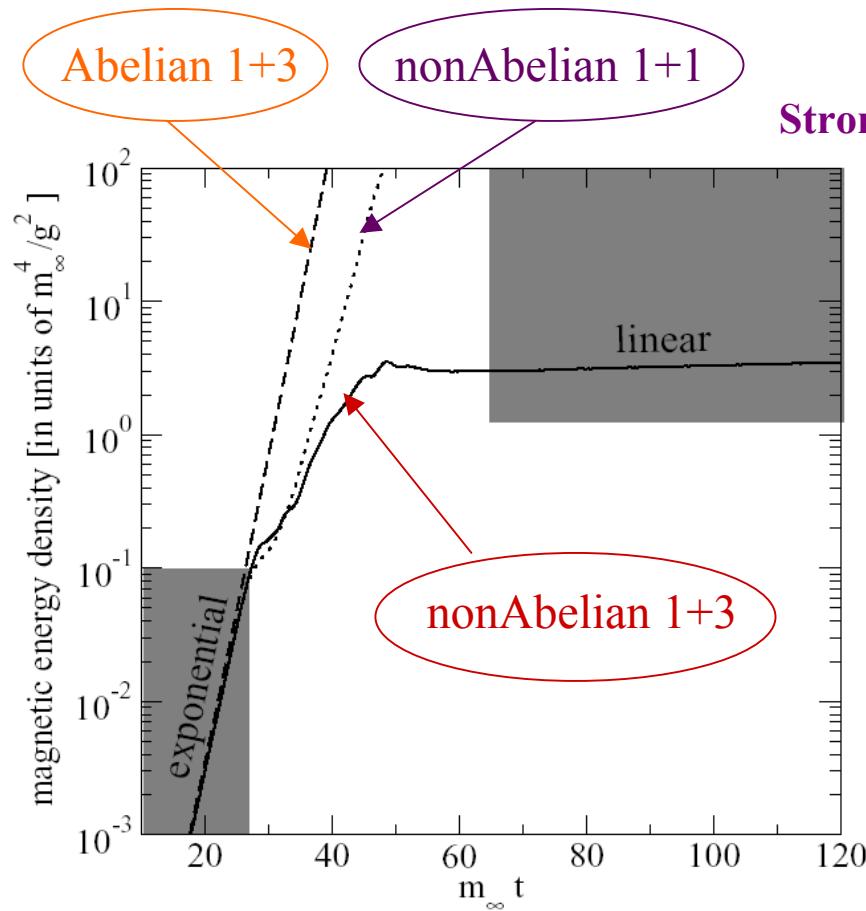
SU(2) Hard Loop Dynamics

Strongly anisotropic particle's momentum distribution

P. Arnold, G.D. Moore & L.G. Yaffe,
hep-ph/0505212
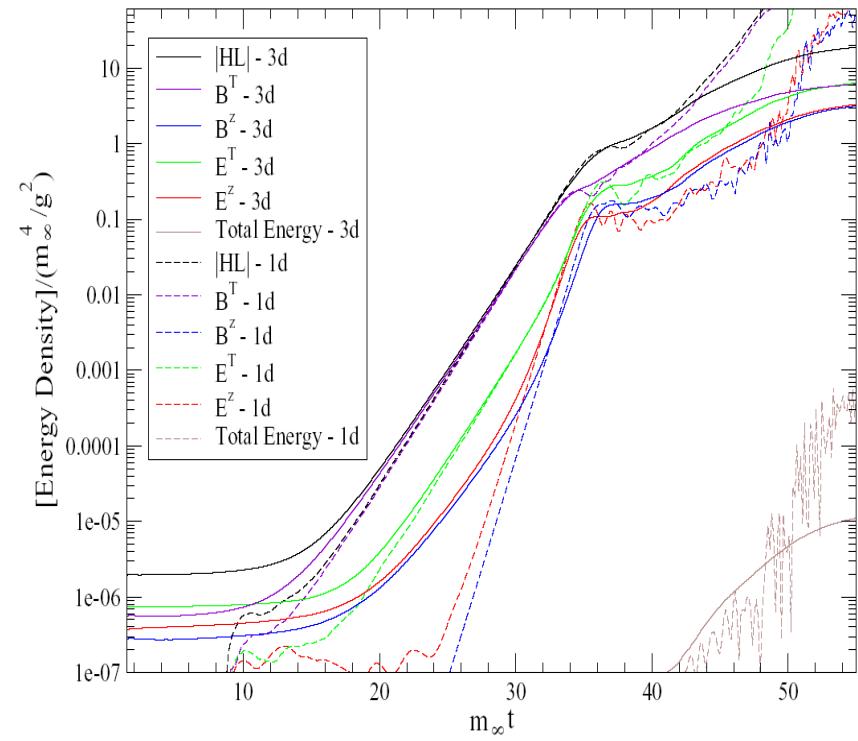
A. Rebhan, P. Romatschke & M.Strickland,
hep-ph/0505261

Abelanization

$$V_{\text{eff}}[\mathbf{A}^a] = -\mu^2 \mathbf{A}^a \cdot \mathbf{A}^a + \frac{1}{4}g^2 f_{abc}f_{ade}(\mathbf{A}^b \cdot \mathbf{A}^d)(\mathbf{A}^c \cdot \mathbf{A}^e)$$

the gauge $\quad A_0^a = 0, \quad A_i^a(t,x,y,z) = A_i^a(x)$

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} = -\frac{1}{2}\mathbf{B}^a \mathbf{B}^a$$

$$= -\frac{1}{4}g^2 f_{abc}f_{ade}(\mathbf{A}^b \cdot \mathbf{A}^d)(\mathbf{A}^c \cdot \mathbf{A}^e)$$

SU(2)

$$\mathbf{B}^a = \nabla \times \mathbf{A}^a + \frac{g}{2}f_{abc}\mathbf{A}^b \times \mathbf{A}^c$$

P. Arnold & J. Lenaghan, Phys. Rev. D **70**, 114007 (2004)
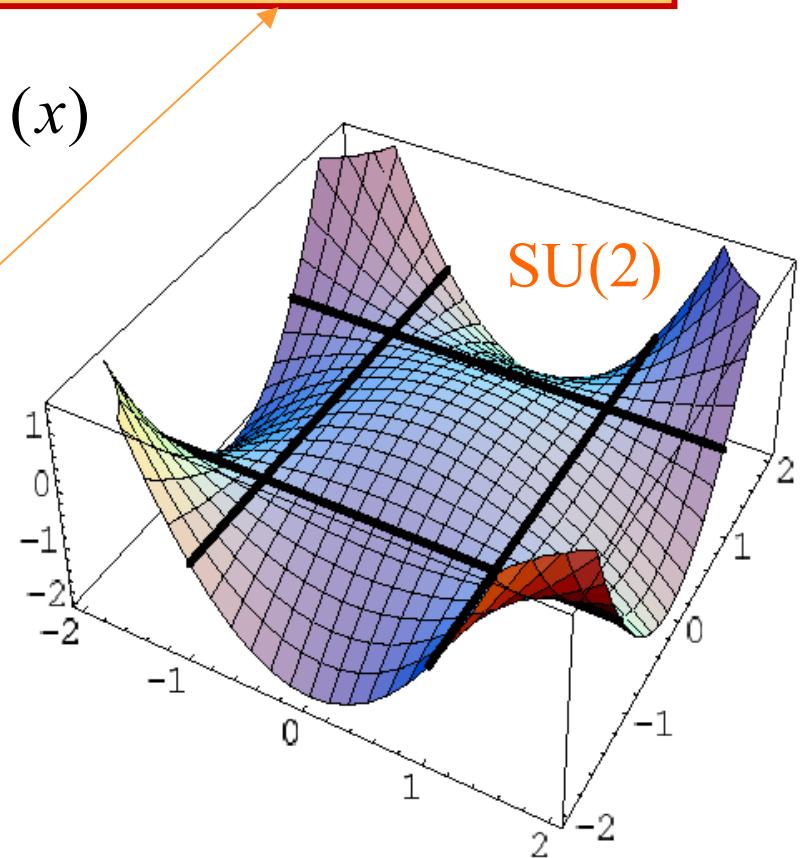
Abelanization – 1+1 numerical simulations

Classical system of colored particles & fields

$$\overline{C} \equiv \int\limits_0^L \frac{dx}{L} \frac{\sqrt{\overline{\mathrm{Tr}((i[A_y, A_z])^2)}}}{\mathrm{Tr}[\mathbf{A}^2]}$$

$$\phi_{\mathrm{rms}} \equiv \sqrt{\int\limits_0^L \frac{dx}{2L} \overline{\mathrm{Tr}[\mathbf{A}^2]}}$$
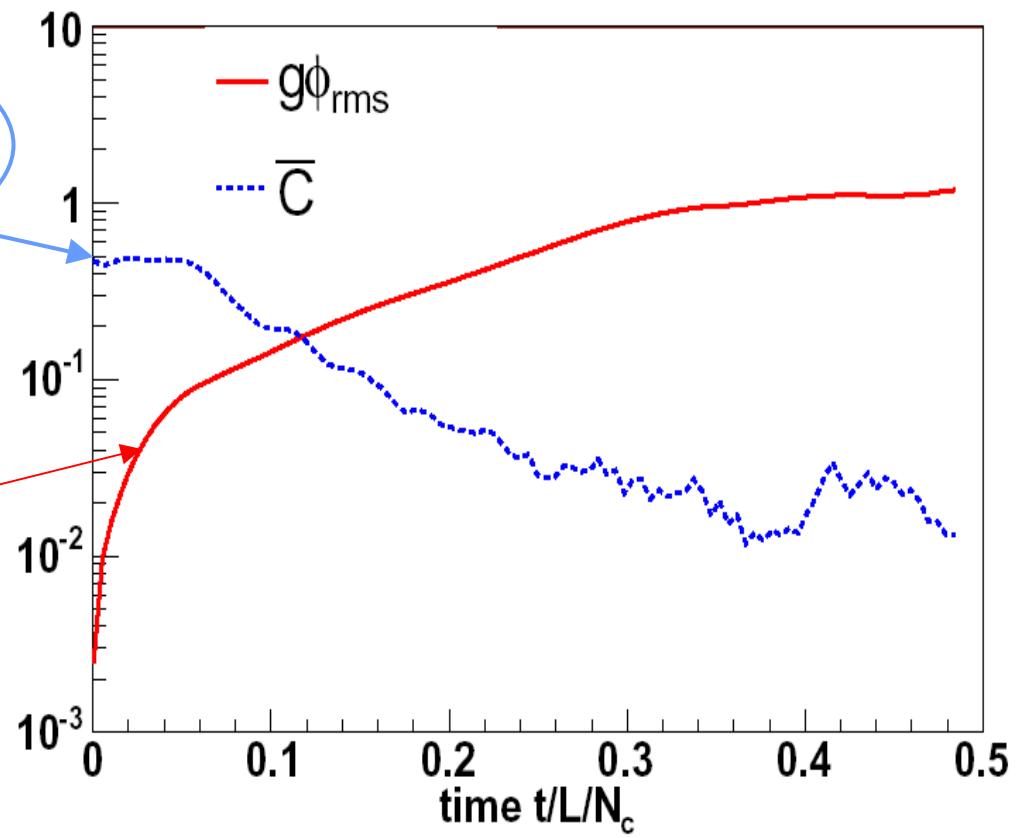
A. Dumitru & Y. Nara, hep-ph/0503121

Abelanization – 1+1 numerical simulations

SU(2) Hard Loop Dynamics

$$\overline{C} \equiv \int_0^L \frac{dz}{L} \frac{\sqrt{\mathrm{Tr}((i[j_x, j_y])^2)}}{\mathrm{Tr}[\mathbf{j}^2]}$$

$$j_{\mathrm{rms}} \equiv \sqrt{\int_0^L \frac{dz}{L} 2\,\mathrm{Tr}[\mathbf{j}^2]}$$

Legend:
- $j_{\mathrm{rms}}/(m_\infty^3/g)$
- $\overline{C}[j]$

x-axis: $\gamma_* \, t$

A. Rebhan, P. Romatschke & M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005)

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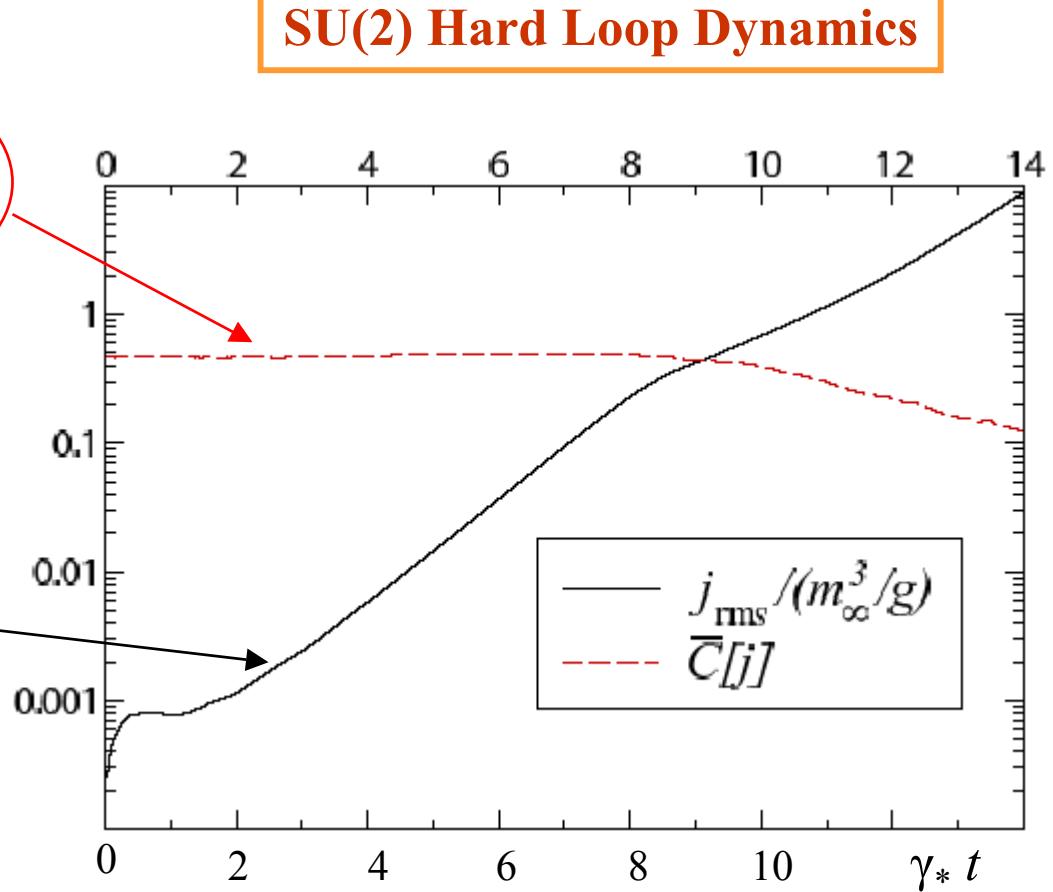
Abelanization – 1+3 numerical simulations

SU(2) Hard Loop Dynamics

$$C \equiv \frac{3}{\sqrt{2}} \frac{\int \frac{d^3 x}{V} \sqrt{\mathrm{Tr}((i[j_x, j_y])^2 + (i[j_y, j_z])^2 + (i[j_z, j_x])^2)}}{\int \frac{d^3 x}{V} \mathrm{Tr}(\mathbf{j}^2)}$$

$$A_i^a \sim e^{\gamma t}$$

$$A_i^a \sim \frac{k_{\text{field}}}{g} << \frac{p_{\text{hard}}}{g}$$

Legend:
- 3+1 non-Abelian
- 1+1 non-Abelian

Axes: C (vertical), $m_\infty t$ (horizontal)

P. Arnold, G.D. Moore & L.G. Yaffe, hep-ph/0505212
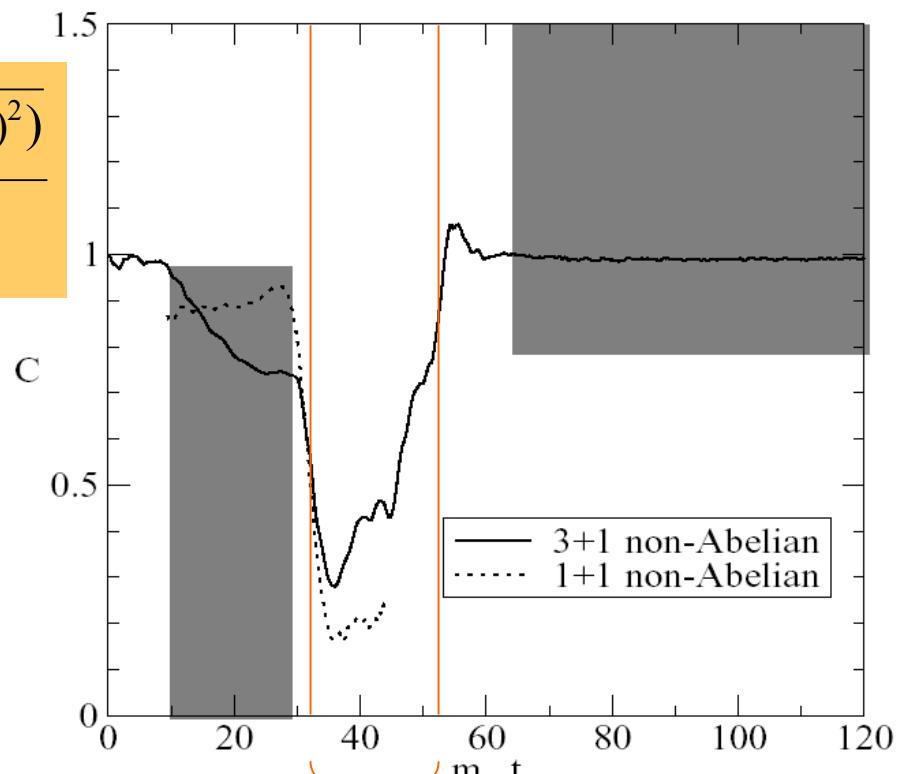
Beyond Hard Loop level

$$V_{\text{eff}}[\mathbf{A}^a] = -\mu^2 \, \text{Tr}[\mathbf{A}^2] + \lambda \, \text{Tr}[\mathbf{A}^4] + \cdots$$

hard-loop term

$V_{\text{eff}}[\mathbf{A}^a]$

$|\mathbf{A}^a|$

$\lambda > 0$

λ depends on the momentum distribution of particles
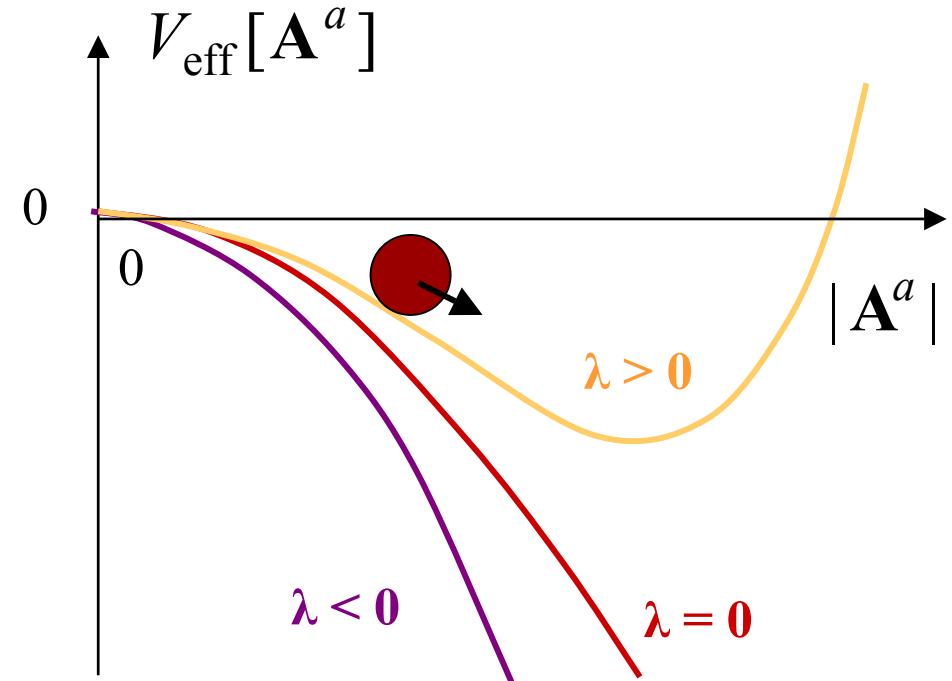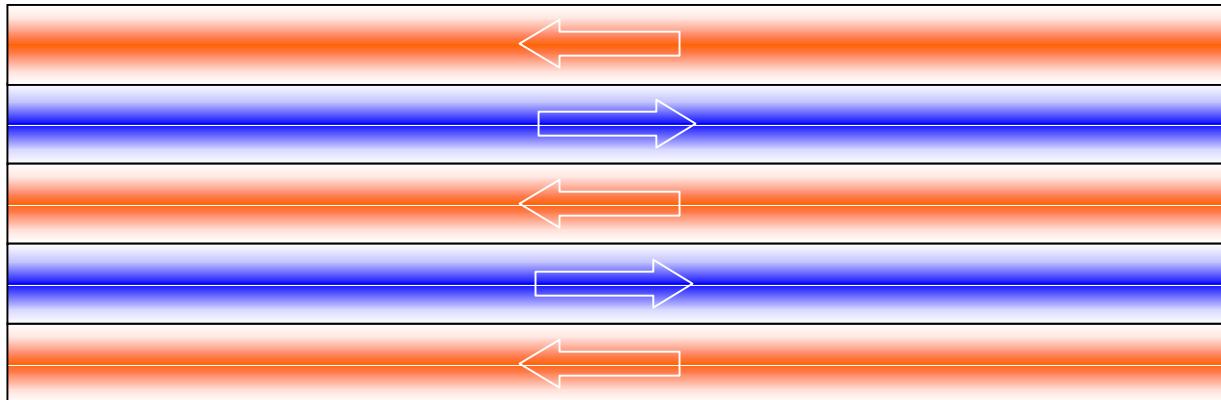
$\lambda < 0$

$\lambda = 0$

Direction of the momentum surplus

j

F

$\otimes$ $\odot$

B
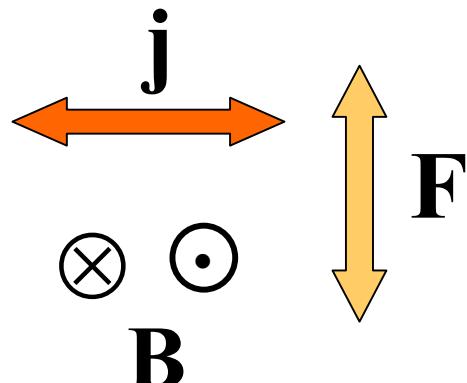
$$\Delta\mathbf{p} = \int dt\,\mathbf{F}$$

Direction of the momentum surplus
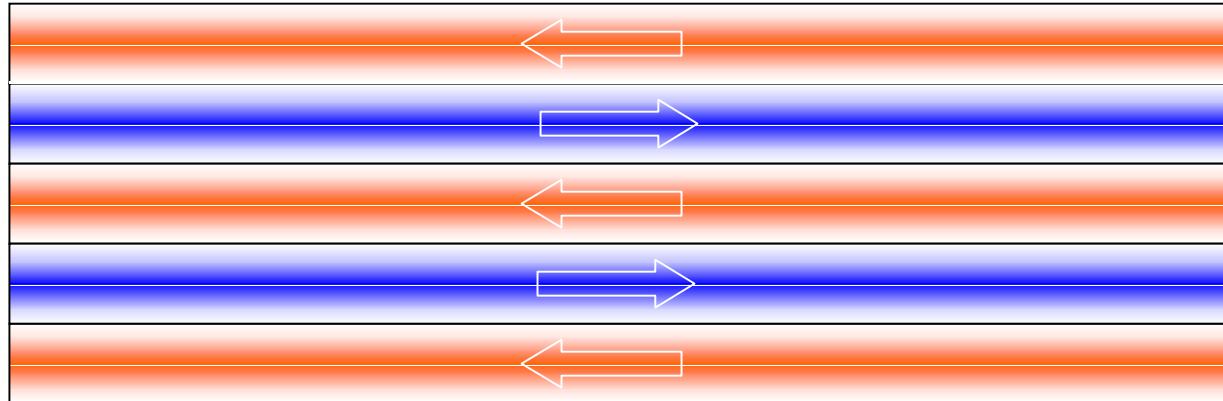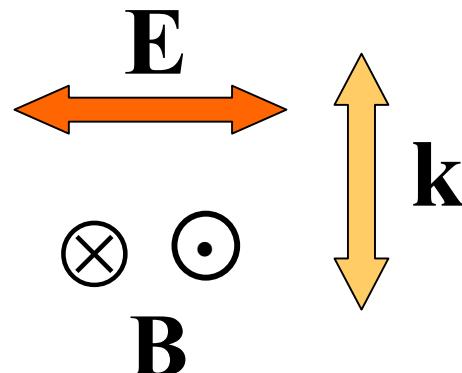
$$\mathbf{P}_{\text{fields}} \sim \mathbf{B}^a \times \mathbf{E}^a \sim \mathbf{k}$$

Isotropization – numerical simulation

Classical system of colored particles & fields

$$T_{ij} = \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j}{E} f(\mathbf{p})$$

Isotropy:

$$T_{xx} = (T_{yy} + T_{zz})/2$$

$(T_{yy}+T_{zz})/2$

T_{xx}

Legend:
- $g^2 T_{xx}$, SU(2)
- $g^2(T_{yy}+T_{zz})/2$, SU(2)
- $g^2 T_{xx}$, U(1)
- $g^2(T_{yy}+T_{zz})/2$, U(1)

y-axis: kinetic pressures
x-axis: time t/L/N_c

A. Dumitru & Y. Nara, hep-ph/0503121

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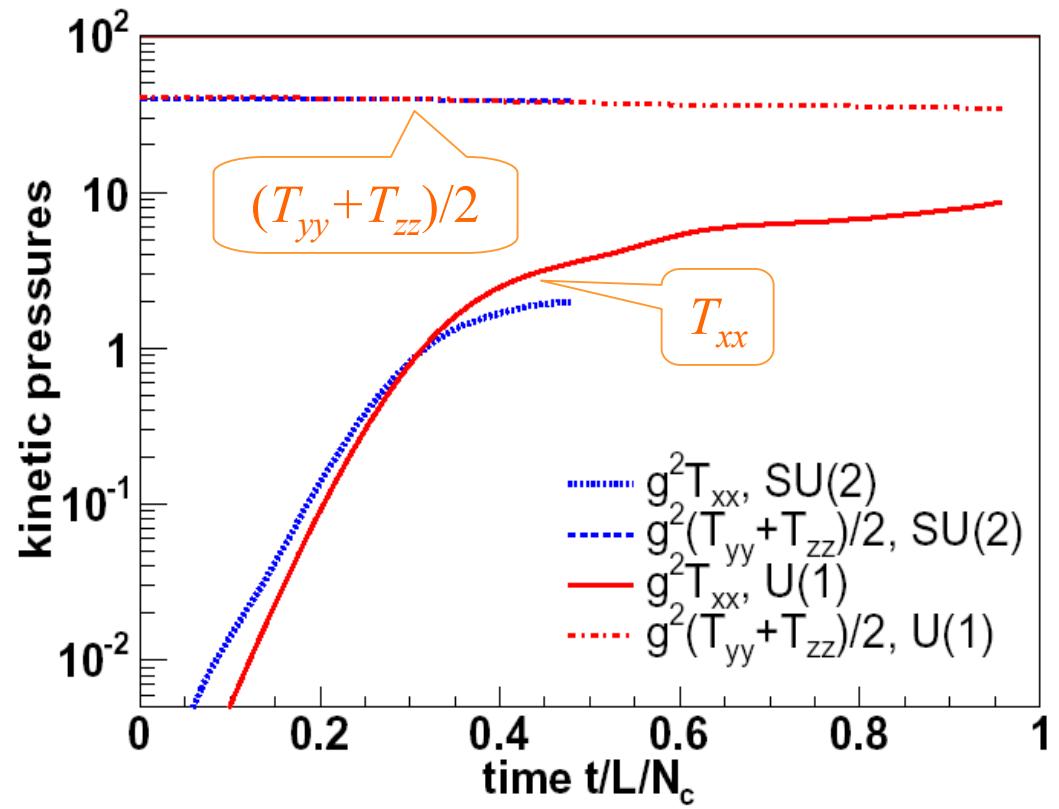
Conclusion

The scenario of instabilities driven equilibration
is a plausible solution of the fast equilibration
problem

Appendix – important contributions

St. Mrówczyński, *Color collective effects at the early stage of ultrarelativistic heavy ion collisions,* Phys. Rev. C **49**, 2191 (1994).

St. Mrówczyński, *Color filamentation in ultrarelativistic heavy-ion collisions,* Phys. Lett. B **393**, 26 (1997).

P. Romatschke and M. Strickland, *Collective modes of an anisotropic quark gluon plasma,* Phys. Rev. D **68**, 036004 (2003)

P. Arnold, J. Lenaghan and G.D. Moore, *QCD plasma instabilities and bottom-up thermalization,* JHEP **0308**, 002 (2003)

Unstable Mode Analysis

Heavy-Ion Phenomenology

J. Randrup and St. Mrówczyński, *Chromodynamic Weibel instabilities in relativistic nuclear collisions,* Phys. Rev. C **68**, 034909 (2003)

P. Arnold, J. Lenaghan, G.D. Moore and L.G. Yaffe, *Apparent thermalization due to plasma instabilities in quark gluon plasma,* Phys. Rev. Lett. **94**, 072302 (2005)

St. Mrówczyński and M. Thoma, *Hard loop approach to anisotropic systems,*
Phys. Rev. D **62**, 036011 (2000)

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